The stochastic block-model and its (variational) inference

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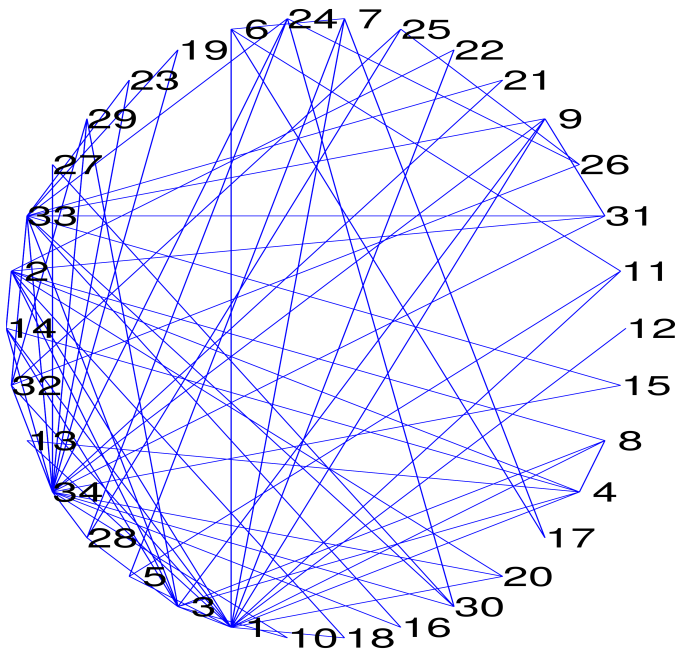
Karate club

$n = 34$ nodes (individuals), link = mutual friendship



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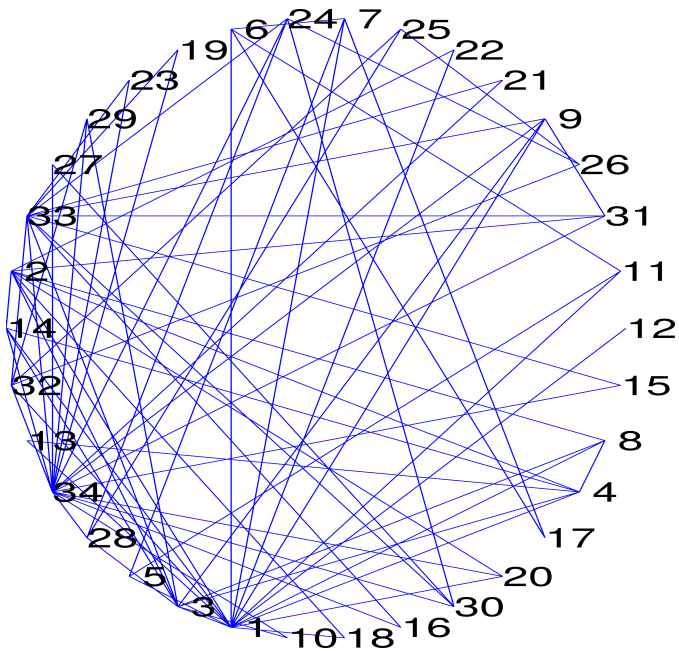

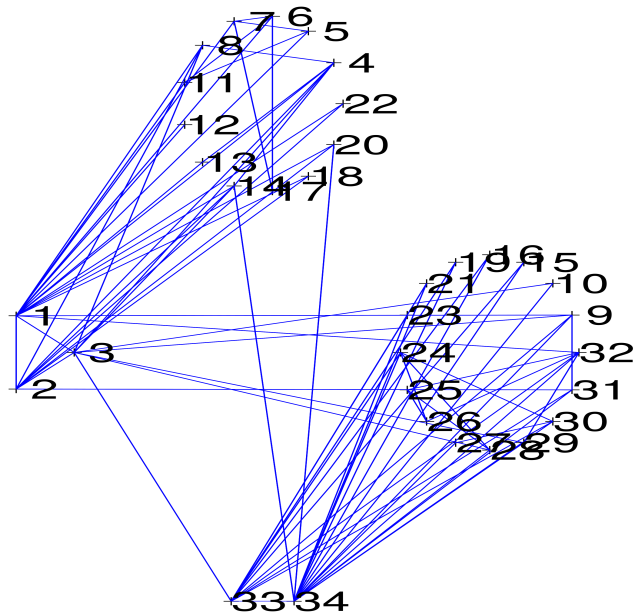

Tree ecological network

- $n = 51$ tree species,
- Y_{ij} = number of shared parasites,
- x_{ij} = (taxonomic, geographic, genetic distances)

Mariadassou & al., Ann. Applied Stat., 2010

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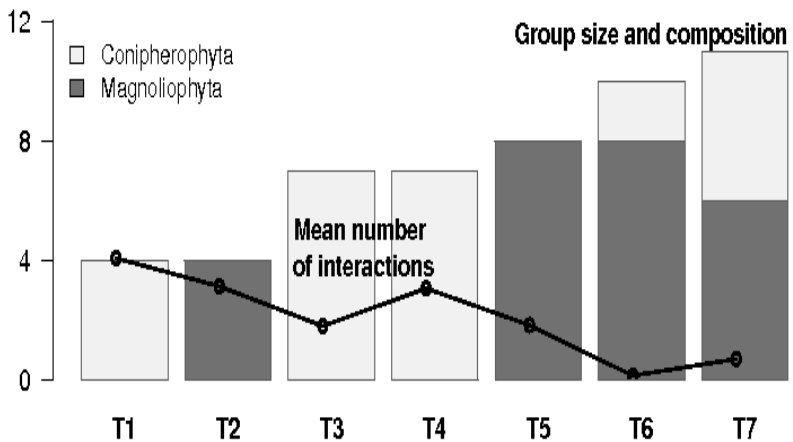
Without covariates: $\mathcal{P}(e^{\gamma_{k\ell}})$



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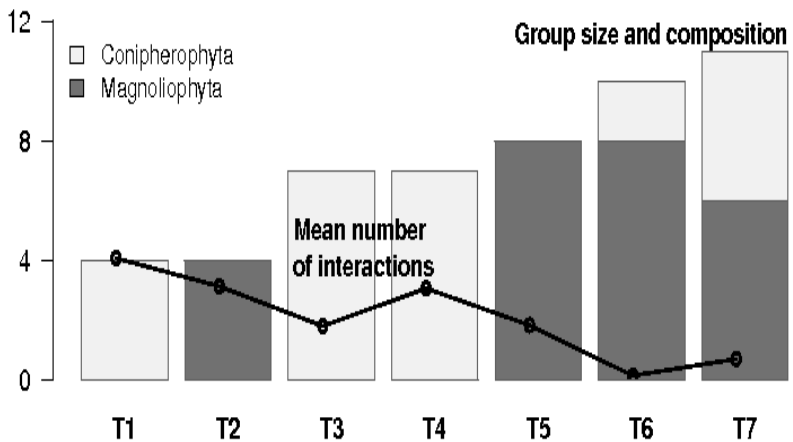
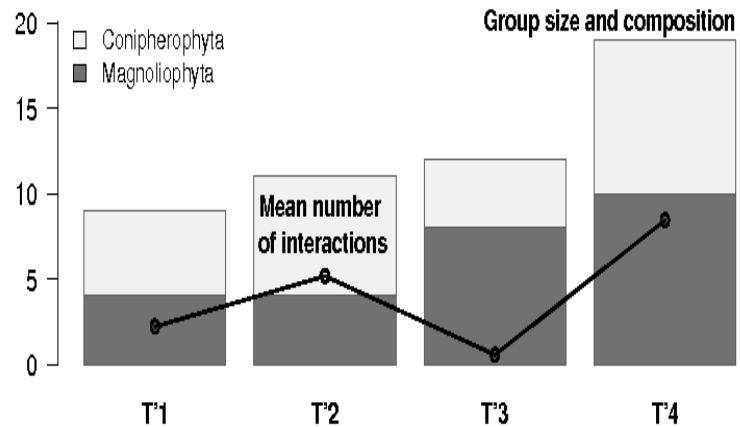
Without covariates: $\mathcal{P}(e^{\gamma_{k\ell}})$

With covariates: $\mathcal{P}(e^{\gamma_{k\ell} + x_{ij}^{\mathsf{T}} \beta})$

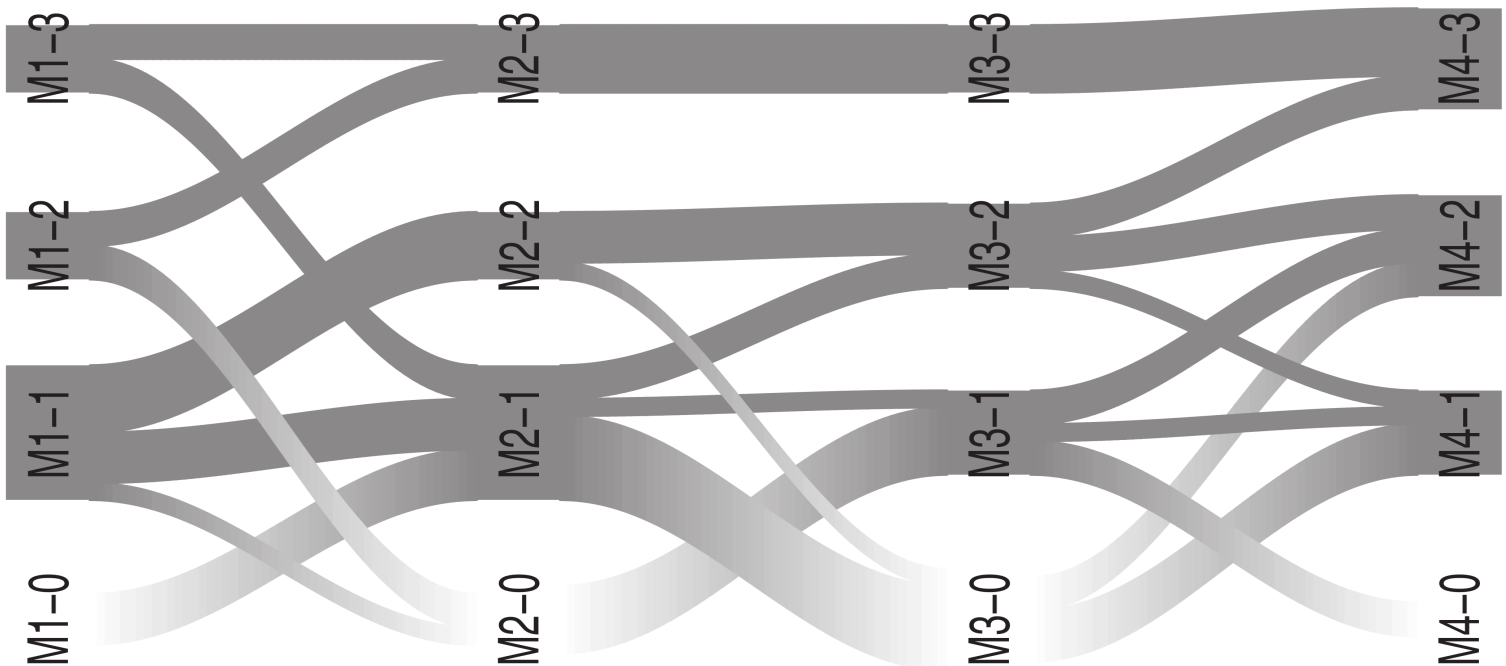




Mariadassou & al., Ann. Applied Stat., 2010

Onager network

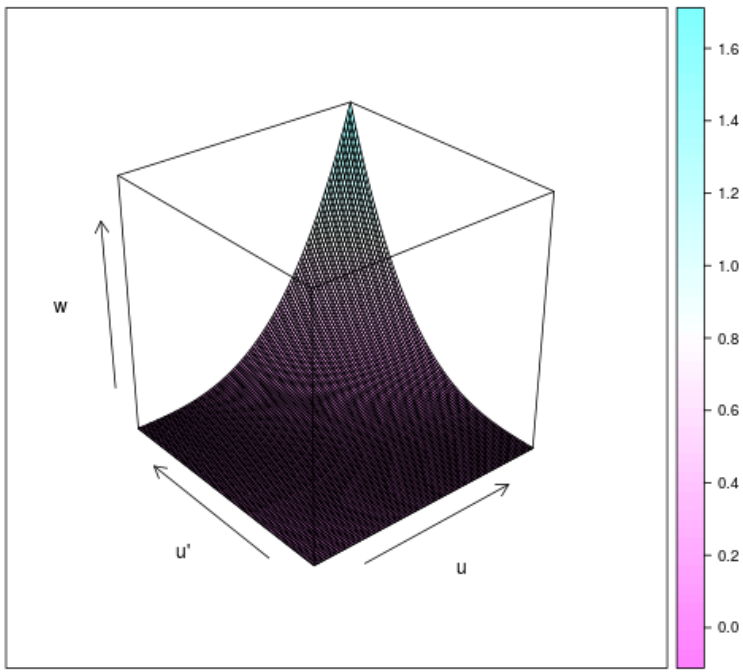
$n = 23$ individuals, $T = 4$ dates, $\widehat{K} = 4$ groups



Matias & Miele, JRSSB, 2017

W-graph

A graphon function: $w(u, u')$.



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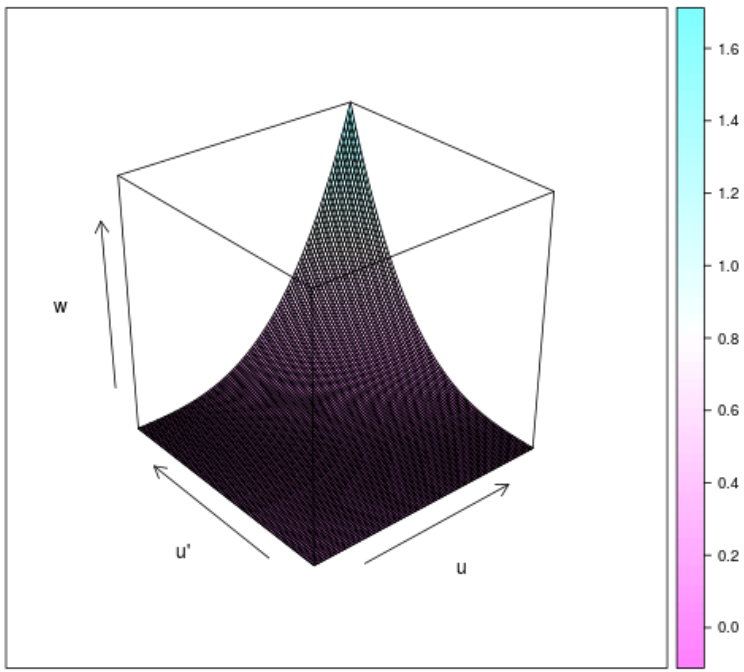
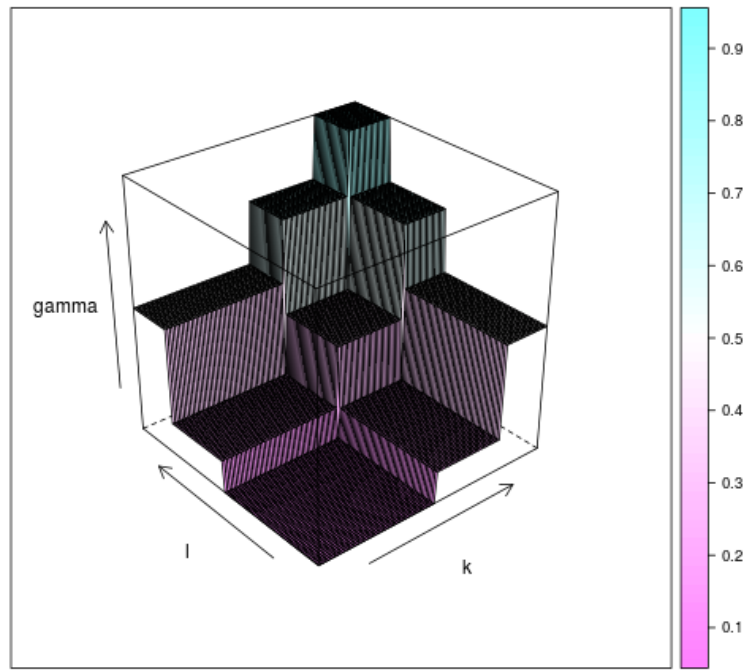


Graphon function of an SBM.



Graphical models (Lauritzen, 96)

Directed graphical models: p faithfull to G (DAG) iff

$$p(U_1, \dots U_m) = \prod_j p(U_j \mid U_{pa_G(j)})$$

where $U_J = \{U_j : j \in J\}$ and $pa_G(j) =$ sets of parents of j in G.

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Undirected graphical models: p faithfull to G iff

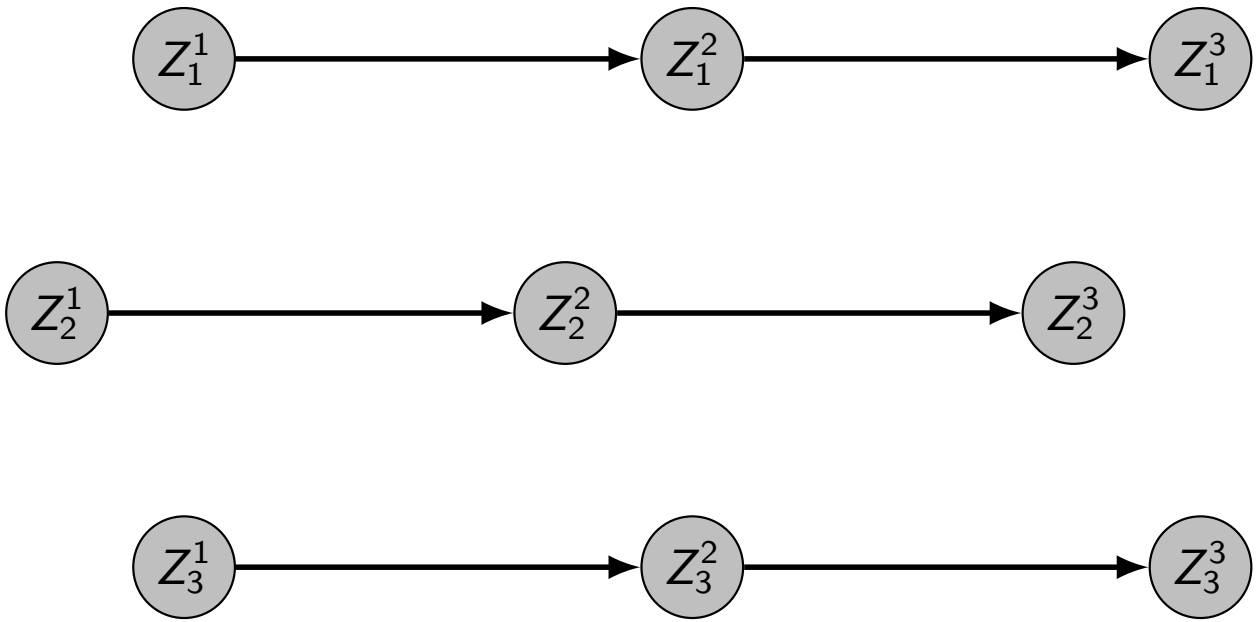
$$p(U_1, \dots U_m) \propto \prod_{C \in \mathcal{C}(G)} \Psi_C(U_C)$$

where $\mathcal{C}(G) =$ set of all maximal cliques of G.

- Directed GM $\rightarrow$ Undirected GM, via moralization.
- Undirected GM: Equivalence between separation and conditional independence.

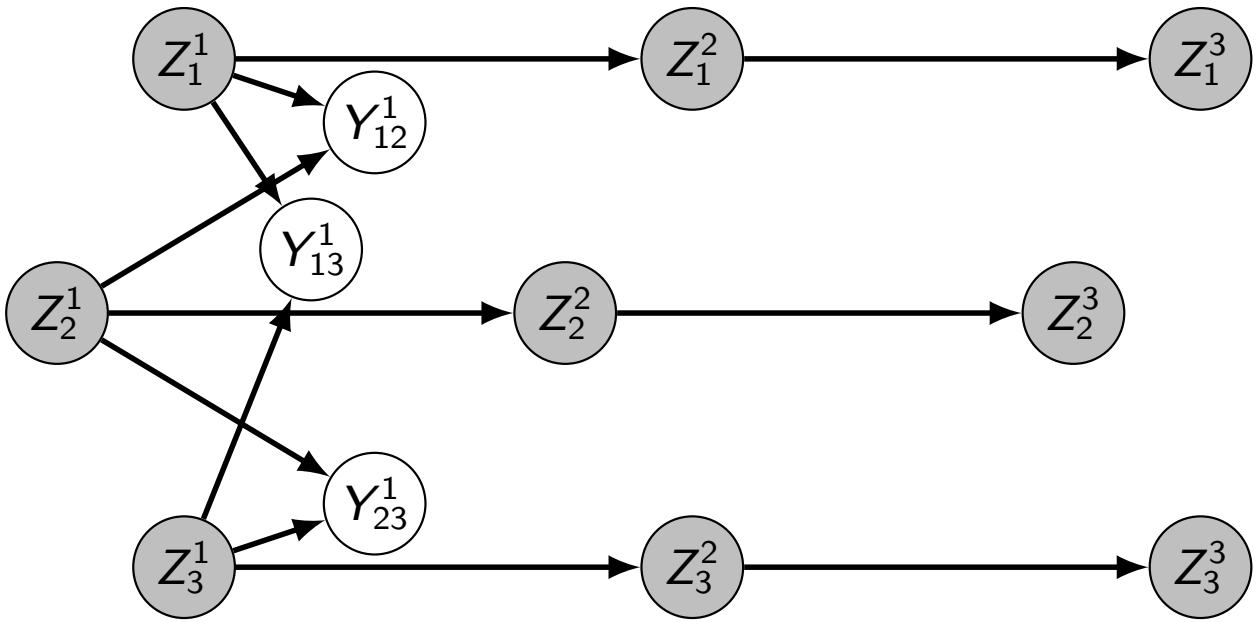
Graphical models for the dynamic SBM

Hidden Markov chains. $n = 3,\ T = 3$



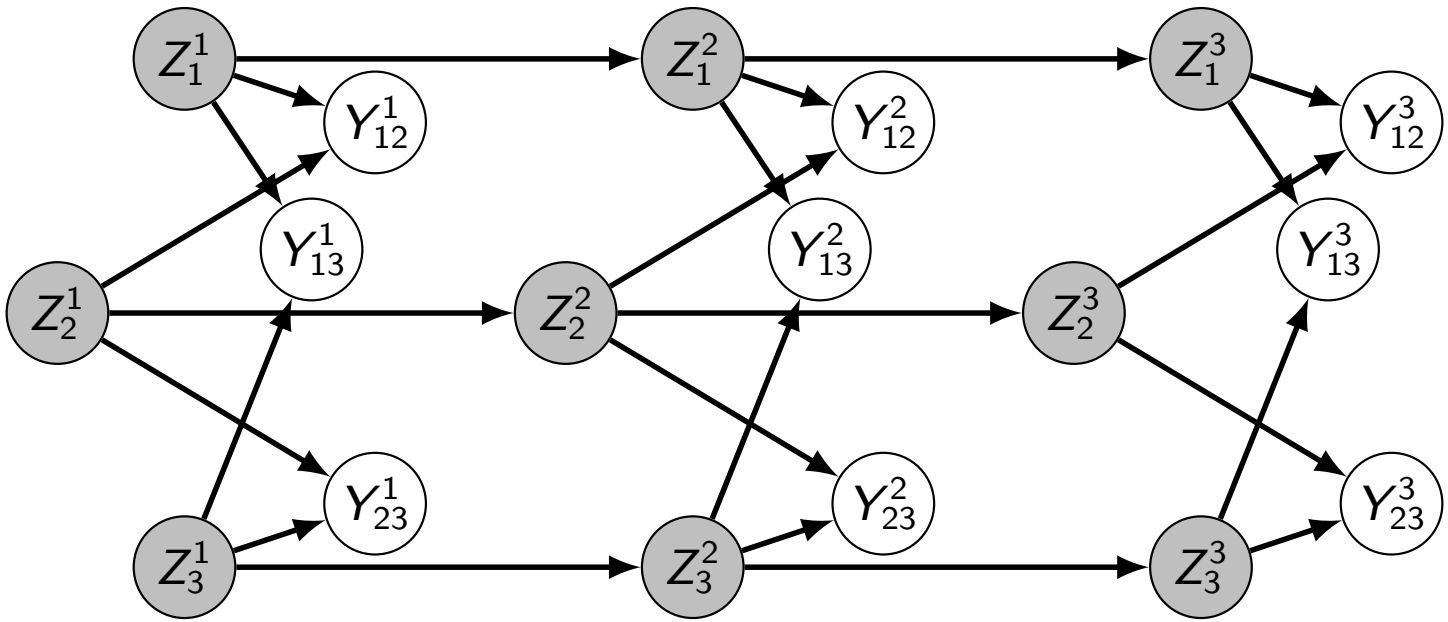
Graphical models for the dynamic SBM

Observed network at $t = 1$.



Graphical models for the dynamic SBM

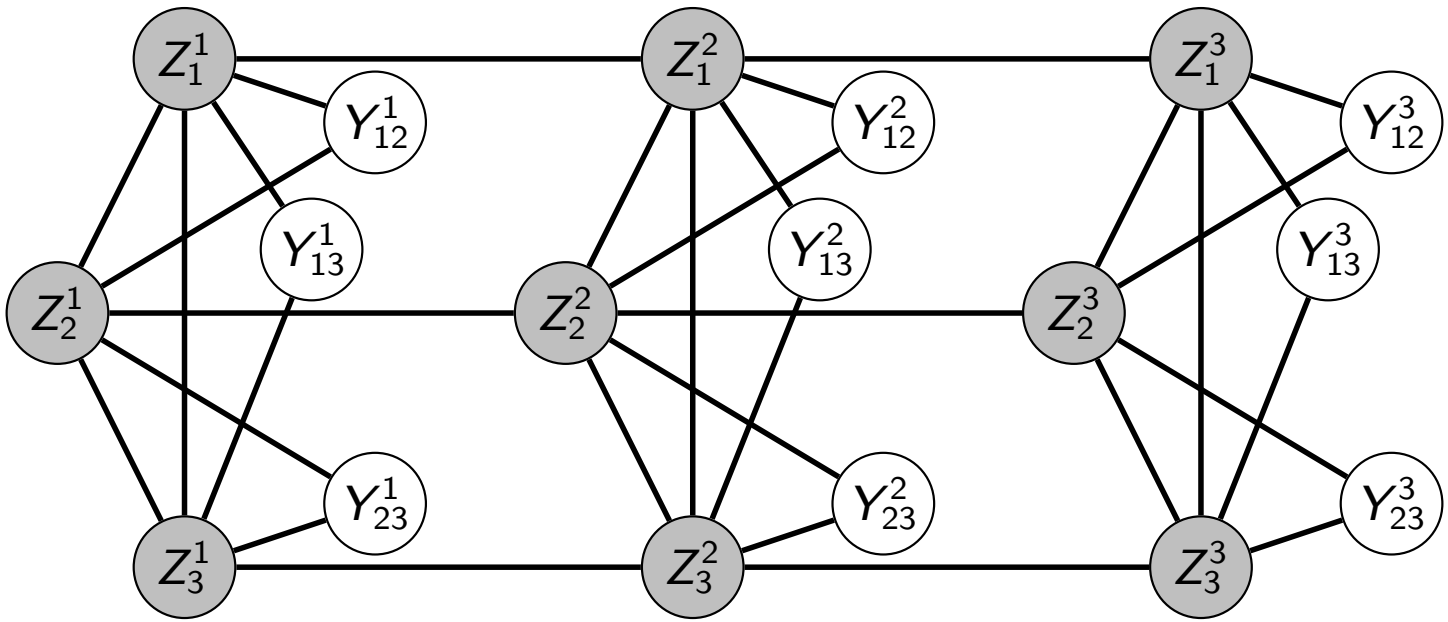
Observed networks at $t = 1, \dots T$.



$\rightarrow$ $(Z^t, Y^t)_t \sim HMM$ with K^n states.

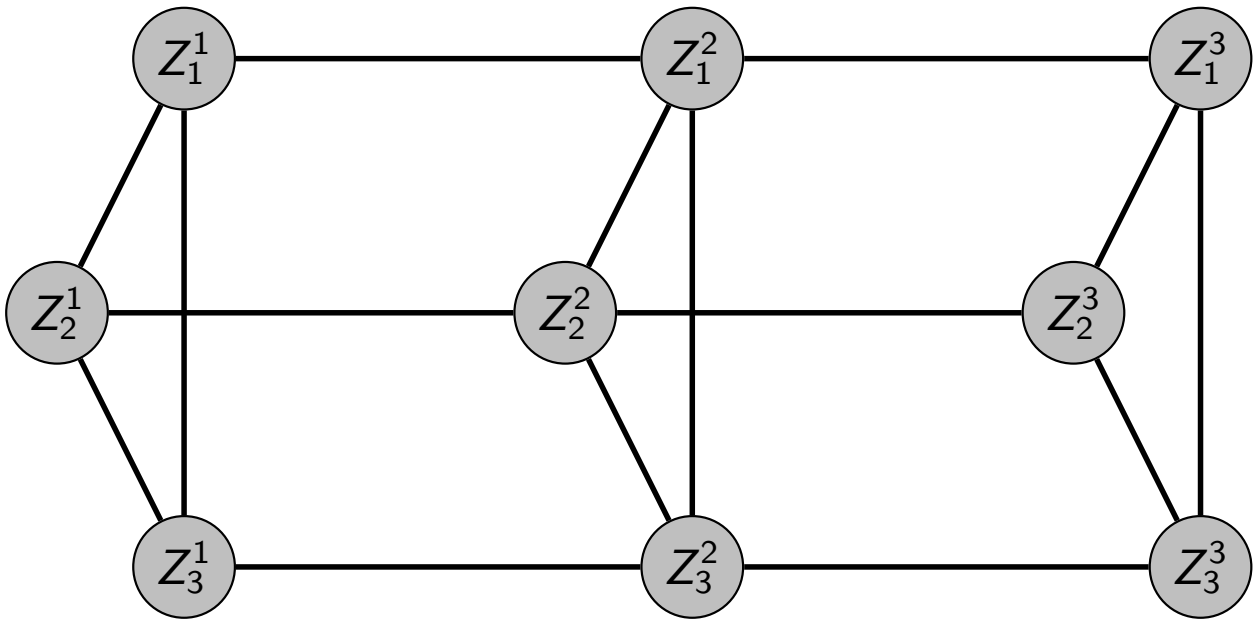
Graphical models for the dynamic SBM

Graph moralization.



Graphical models for the dynamic SBM

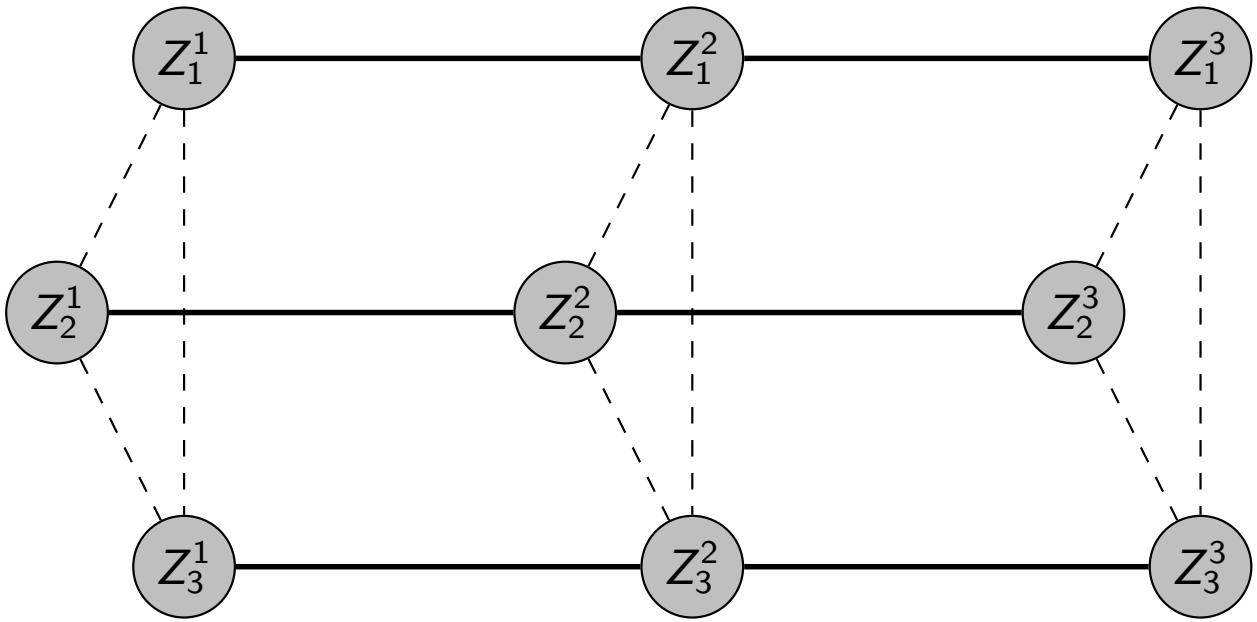
Graphical model of $p(Z \mid Y)$.



$\rightarrow$ (Heterogeneous) Markov chain with K^n states

Graphical models for the dynamic SBM

Variational approximation: $p(Z \mid Y) \simeq \prod_i q_i(Z_i) \neq \prod_{i,t} q_{it}(Z_i^t)$

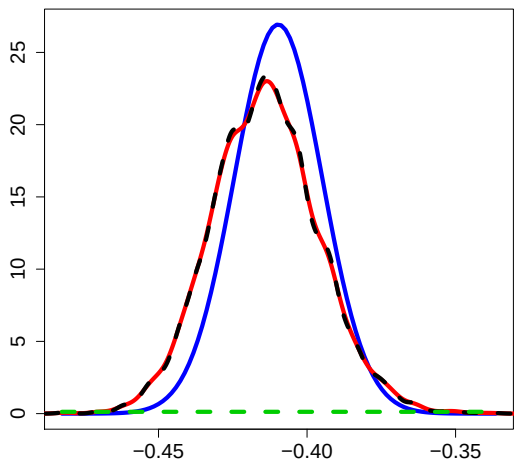


$\rightarrow$ Partial mean-field approximation

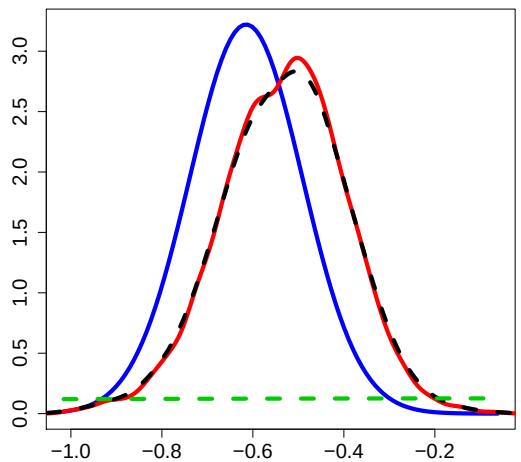
Tree interaction network

$n = 51$ tree species, Y_{ij} = number of shares parasites, Poisson emission, 3 covariates

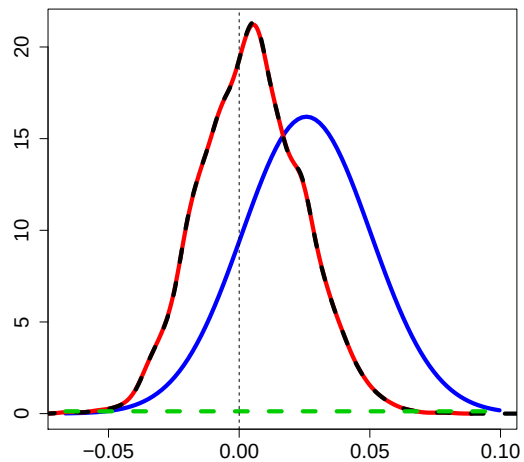
taxonomy



geography

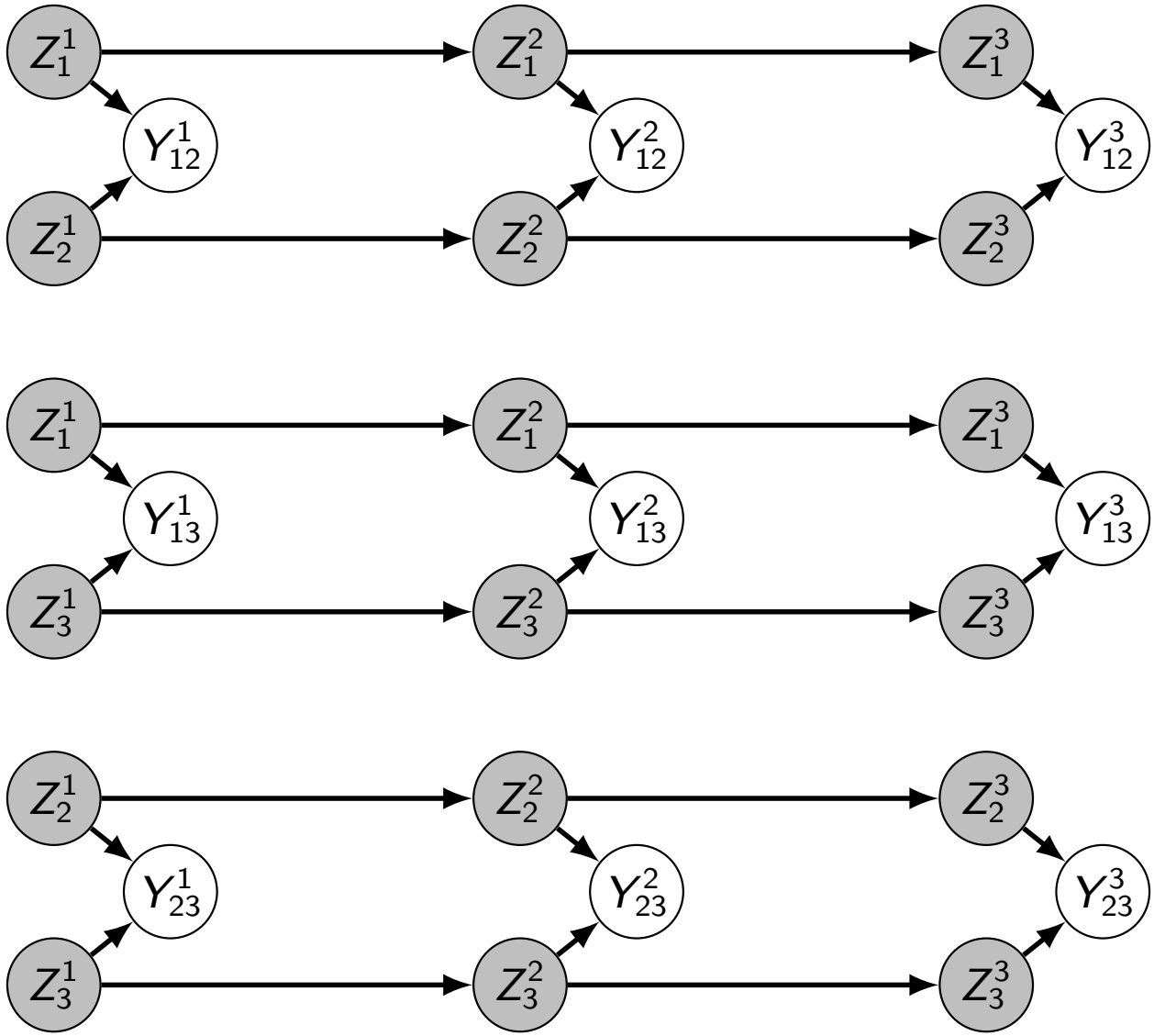


genetics



$p(\beta)$, $\widetilde{p}(\beta \mid \widehat{K})$, $\widehat{p}(\beta \mid Y, \widehat{K})$, $\widehat{p}(\beta \mid Y) = \sum_K \widehat{p}(K \mid Y)\widehat{p}(\beta \mid Y, K)$

Composite likelihood for dynamic SBM



$\rightarrow$ $n(n-1)/2$ HMMs with K^2 states.

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